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An alternate relaxation approximation to conservation laws

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Abstract

In this paper we introduce a slight modification to the relaxation system of Jin and Xin which approximates a conservation law. The proposed alternate system satisfies an integral constraint that is more consistent than the standard one while retaining the semilinear structure. We establish L^∞ estimates under the usual subcharacteristic condition and also construct a convex entropy.

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1. Introduction

Consider the scalar conservation law

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$ and u_0, f are some given smooth functions. Jin and Xin [2] have proposed the system

$$\begin{cases} \partial_t u_\varepsilon + \partial_x v_\varepsilon = 0, \\ \partial_t v_\varepsilon + a^2 \partial_x u_\varepsilon = -\dfrac{1}{\varepsilon}(v_\varepsilon - f(u_\varepsilon)), \end{cases} \tag{1.2}$$

with initial conditions

$$\begin{cases} u_\varepsilon(x,0) = u_0, \\ v_\varepsilon(x,0) = f(u_0), \end{cases} \tag{1.3}$$

as a new way of regularizing (1.1). Using the Chapman–Enskog Expansion one obtains, formally,

$$v_\varepsilon = f(u_\varepsilon) + \varepsilon v_1^\varepsilon + \mathrm{O}(\varepsilon^2), \tag{1.4}$$

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where

$$\begin{cases} v_1^\varepsilon = \beta(u_\varepsilon)\partial_x u_\varepsilon, \\ \beta(u) = a^2 - [f'(u)]^2. \end{cases} \tag{1.5}$$

Introducing (1.4) in (1.1) we obtain, up to $O(\varepsilon)$

$$\partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \varepsilon\partial_x(\beta(u_\varepsilon)\partial_x u_\varepsilon). \tag{1.6}$$

Now (1.6) is parabolic if

$$|f'(u)|^2 \leqslant a^2, \tag{1.7}$$

which is known as the Liu's subcharacteristic condition. Natalani [4] has proved that solutions to (1.2) converge strongly to the unique entropy solution of (1.1) as $\varepsilon \to 0$.

The aim of the present work is to propose an alternate for (1.2), our proposal is

$$\begin{cases} \partial_t u_\varepsilon + \partial_x v_\varepsilon = v_\varepsilon - f(u_\varepsilon), \\ \partial_t v_\varepsilon + a^2\partial_x u_\varepsilon = -\dfrac{1}{\varepsilon}(v_\varepsilon - f(u_\varepsilon)), \end{cases} \tag{1.8}$$

with initial conditions (1.3). Proceeding as in (1.4) we obtain, instead of (1.6),

$$\partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \varepsilon\partial_x(\beta(u_\varepsilon)\partial_x u_\varepsilon) - \varepsilon\beta(u_\varepsilon)\partial_x u_\varepsilon, \tag{1.9}$$

where β is as in (1.5). The condition (1.7) is once again enough to ensure parabolicity of (1.9). A nice feature of (1.8) is that it satisfies the conservation law

$$\partial_t(u_\varepsilon + \varepsilon v_\varepsilon) + \partial_x(v_\varepsilon + \varepsilon a^2 u_\varepsilon) = 0, \tag{1.10}$$

leading to

$$\int u_\varepsilon(\cdot, t) + \varepsilon v_\varepsilon(\cdot, t) = \int u_0(\cdot) + \varepsilon f(u_0(\cdot)), \tag{1.11}$$

for all t. In contrast the integral invariant for (1.2) is

$$\int u_\varepsilon(\cdot, t) = \int u_0(\cdot).$$

Note that systems like (1.8) with a ε^{-1} factor also in the first equation have been considered by Katsoulakis and Tzavaras [3]. In fact such systems are obtained when one diagonalizes (1.2) with the help of Riemann invariants. This is possible only for one-dimensional conservation laws. Numerical results for this alternate approximation (1.8) will be reported elsewhere.

Our results and techniques are very similar to Natalani [4] and Jin [1]. In the former it was shown that the system (1.2) converges to the entropy solution of (1.1). To achieve that it was shown that there exists a constant α_0, depending only on the L^∞ norm of the initial data and on the flux function f, such that for $\alpha \geqslant \alpha_0$ the solutions of (1.2) are bounded independent of ε. The crucial fact that was used is the quasimonotone nature of (1.2) under the condition (1.7). We follow the same procedure in Section 2 for the system (1.8) and establish L^∞ bounds. In Section 3 following Jin [1] we construct a convex entropy.

2. L^∞ estimates

In this section we shall establish uniform supremum norm bound for the solutions $(u_\varepsilon, v_\varepsilon)$ of the Cauchy problem (1.8) with data (1.3). We shall assume that

$$(\mathrm{H}) \quad \begin{cases} f \text{ is uniformly Lipschitz with} \\ -a < f'(u) < a \\ \text{for all } u \text{ and } f(0) = 0. \end{cases}$$

Further, we set

$$N_0 = \max\{\|u_0\|_\infty, \|v_0\|_\infty\}. \tag{2.1}$$

Theorem 2.1. *Assume* (H) *then for any* $N_0 > 0$, $\varepsilon > 0$ *with* $a\varepsilon < 1$ *there exists an unique, globally bounded solution* $(u^\varepsilon, v^\varepsilon)$ *to* (1.8) *and* (1.3) *in* $C[0, \infty); L^1_{\text{loc}}(\mathbb{R}^2))$ *with*

$$\|u_\varepsilon \pm a v_\varepsilon\|_{L^\infty(\mathbb{R}\times(0,\infty))} \leqslant 2N_0(1+a)/(1-a\varepsilon). \tag{2.2}$$

Proof. Our proof is based on Natalani [4]. We first diagonalize (1.8), dropping the index ε we set

$$r_+ = v - au, \quad r_- = -v - au. \tag{2.3}$$

These are the Riemann invariants with respect to the characteristic fields $\lambda_\pm = \pm a$ and satisfy

$$\begin{cases} \partial_t r_+ - a\partial_x r_+ = -\left(a + \dfrac{1}{\varepsilon}\right) G(r), \\ \partial_t r_- + a\partial_x r_- = -\left(a - \dfrac{1}{\varepsilon}\right) G(r), \end{cases} \tag{2.4}$$

where

$$G(r) = \frac{r_+ - r_-}{2} - f\left(-\frac{r_+ + r_-}{2a}\right), \tag{2.5}$$

with the initial conditions

$$\begin{cases} r_+(x,0) = v_0(x) - au_0(x), \\ r_-(x,0) = -v_0(x) - au_0(x). \end{cases} \tag{2.6}$$

Now (2.4) is quasimonotone because $a\varepsilon < 1$. The existence of a local solution on some interval $\mathbb{R} \times (0, \bar{T})$, $\bar{T} > 0$ is established in exactly similar manner as in Natalini [4]. It is therefore suffices to establish the bound (2.2). For this consider the system of ordinary differential equations

$$\begin{cases} \dot{p}_+ = -\left(a + \dfrac{1}{\varepsilon}\right) G(p), \\ \dot{p}_- = -\left(a - \dfrac{1}{\varepsilon}\right) G(p), \end{cases} \tag{2.7}$$

with

$$p_+(0) = p_-(0) = R_0. \tag{2.8}$$

Changing variables,

$$U = -\frac{p_+ + p_-}{2a}, \quad v = \frac{p_+ - p_-}{2}, \tag{2.9}$$

(2.7)–(2.8) is transformed to

$$\begin{cases} \dot{U} = V - f(U), \\ \dot{V} = -\dfrac{1}{\varepsilon}(V - f(U)), \end{cases} \tag{2.10}$$

with

$$U(0) = -R_0/a, \quad V(0) = 0.$$

This can be rewritten in the integral formulation

$$\begin{cases} U(t) = -\dfrac{R_0}{a} - \int_0^t \mathrm{e}^{-(t-s)/\varepsilon} f(U(s))\,\mathrm{d}s, \\ V(t) = \dfrac{1}{\varepsilon} \int_0^t \mathrm{e}^{-(t-s)/\varepsilon} f(U(s))\,\mathrm{d}s. \end{cases} \tag{2.11}$$

Using the hypothesis (H) we obtain

$$|U(t)| \leqslant \frac{1}{a}|R_0| + a \int_0^t \mathrm{e}^{-(t-s)/\varepsilon} |U(s)|\,\mathrm{d}s,$$

an application of Gronwall's inequality (Walter [5, p. 14]) yields

$$|U(t)| \leqslant \frac{R_0}{a(1-a\varepsilon)} \left[1 - a\varepsilon \mathrm{e}^{(a-1/\varepsilon)t}\right],$$

and hence

$$|U(t)| \leqslant \frac{R_0}{a(1-a\varepsilon)}. \tag{2.12}$$

Similarly, we obtain

$$|V(t) \leqslant \frac{R_0}{(1-a\varepsilon)}. \tag{2.13}$$

Going back to $p_\pm$ (cf (2.9)) we obtain

$$\frac{-2R_0}{(1-a\varepsilon)} \leqslant p_\pm(t) \leqslant \frac{2R_0}{1-a\varepsilon}. \tag{2.14}$$

The initial condition (2.6) satisfy

$$|r_\pm(x,0)| \leqslant (1+a)N_0,$$

where N_0 is defined in (2.1).

We now choose $R_0 = (1+a)N_0$ and denote $(p_+^\pm, p_-^\pm)$ the solutions corresponding to

$$R_0^\pm = \pm(1+a)N_0,$$

with this choice we have

$$p_\pm^-(0) \leqslant r_\pm(x,0) \leqslant p_\pm^+(0),$$

and hence

$$p_\pm^-(t) \leqslant r_\pm(x,t) \leqslant p_\pm^+(t),$$

which in turn yields

$$|r_\pm(x,t)| = |v_\varepsilon(x,t) \pm a u_\varepsilon(x,t)| \leqslant 2N_0 \,\frac{(1+a)}{(1-a\varepsilon)}.$$

From (2.3) we have

$$v = \frac{r_+ - r_-}{2}, \quad u = -\frac{r_+ + r_-}{2},$$

from which we obtain

$$|v^\varepsilon(x,t)| \leqslant 2N_0(1+a)/(1-a\varepsilon),$$

$$|u^\varepsilon(x,t)| \leqslant 2N_0(1+a)/a(1-a\varepsilon). \qquad \square$$

3. A convex entropy

Jin [1] has shown that the function

$$\mathbb{H}(u, v) = \frac{a}{2} u^2 + \psi(u, v), \tag{3.1}$$

is an entropy for the system (1.2). Here

$$\psi(u, v) = \frac{1}{2a} v^2 + \phi(v - au), \tag{3.2}$$

satisfying

$$\psi_v|_{v=f(u)} = 0, \tag{3.3}$$

where ϕ is the solution to the ODE

$$\phi'(\eta) = -f(h^{-1}(\eta))/a, \tag{3.4}$$

with h^{-1} being the inverse function of

$$h(u) = f(u) - au. \tag{3.5}$$

The entropy flux is

$$\mathbb{J}(u, v) = -a\phi(v - au) + auv. \tag{3.6}$$

Here $(u, v) \in \Omega$ with

$$\Omega = \{(u, v)|v = f(\overline{u}) + a(u - \overline{u}), (u, \overline{u}) \in I \times I\}, \tag{3.7}$$

where the interval I in R contains 0 as an interior point. The main result of Jin [1] is the entropy inequality

$$\mathbb{H}_t + \mathbb{J}_x = \frac{-1}{\varepsilon} (v - f(u))\psi_v \leqslant 0, \tag{3.8}$$

furthermore, $\mathbb{H}(u, v)$ is a convex function under the subcharacteristic condition (1.7). For the system (1.8) we obtain

$$\mathbb{H}_t + \mathbb{J}_x = -\frac{1}{\varepsilon} (v - f(u))[\psi_v - \varepsilon\{\psi_u + au\}], \tag{3.9}$$

which is shown to be nonpositive in the lemma below. Before that we define for $(u, v) \in \Omega$

$$\begin{cases} \alpha = \sup\limits_{u \neq \tilde{u}} \left| \dfrac{f(\tilde{u} + au)}{u - \tilde{u}} \right|, \\ \tilde{u} = h^{-1}(v - au). \end{cases}$$

Lemma 3.1. *Assume that* $0 < \varepsilon < \varepsilon_0 < 1/\alpha$ *then* (3.8) *is nonpositive*.

Proof. Set

$$q = -\frac{1}{\varepsilon} (v - f(u))$$

and multiply the equations in (1.8) by ψ_u and ψ_v and sum them to obtain

$$\psi_t + \psi_u v_x + a^2\psi_v u_x = -\varepsilon q\psi_u + q\psi_v.$$

Since $\psi_u = v - a\psi_v$ (follows from (3.2)) we have

$$\psi_t + vv_x - a\psi_v[v_x - au_x] = -\varepsilon q\psi_u + q\psi_v.$$

Using $\psi_v = v/a + \phi'$ we obtain

$$\psi_t + avu_x - \phi_x = -\varepsilon q\psi_u + q\psi_v,$$

i.e.,

$$\psi_t + a(uv)_x - auv_x - \varphi_x = -\varepsilon q\psi_u + q\psi_v.$$

Using the first equation in (1.8), $uu_t + uv_x = -\varepsilon uq$, we have

$$\psi_t + a(uv)_x - a\{-\varepsilon uq - uu_t\} - \varphi_x = -\varepsilon q\psi_u + q\psi_v. \tag{3.10}$$

As in Jin [1], from (3.2)–(3.3) we obtain

$$\psi_v|_{v=f(u)} = f(u)/a + \phi'(h(u)) = 0, \tag{3.11}$$

where $h(u)$ is defined in (3.5). Now, from (1.7), $h'(u) = f'(u) - a < 0$ and by Implicit function theorem, $h(u)$ is invertible and therefore for given $(u, v) \in \Omega$, there exists a $\tilde{u} \neq u$ such that

$$h(\tilde{u}) = v - au, \tag{3.12}$$

and from (3.11)

$$\begin{aligned}\psi_u &= -a\phi'(v - au) = -a\phi'(h(\tilde{u})) = f(\tilde{u}),\\ \psi_v &= v/a + \phi'(v - au) = \frac{1}{a}[au + h(\tilde{u})] - f(\tilde{u})/a,\end{aligned} \tag{3.13}$$

and from (3.5)

$$\psi_v = u - \tilde{u}. \tag{3.14}$$

The right-hand side of (3.8) can be written as

$$\varepsilon q\psi_v\left[\frac{1}{\varepsilon} - \left\{\frac{f(\tilde{u}) + au}{u - \tilde{u}}\right\}\right]. \tag{3.15}$$

From the hypothesis

$$\left|\frac{f(\tilde{u}) + au}{u - \tilde{u}}\right| \leqslant \alpha < \frac{1}{\varepsilon_0} < \frac{1}{\varepsilon}.$$

This implies that the quantity in the square bracket in (3.13) is positive and consequently it suffices to show that $q\psi_v \leqslant 0$. This has been shown by Jin [1] in his Lemma 2.1 as follows. One needs to show

$$-\frac{1}{\varepsilon}(v - f(u))(u - \tilde{u}) \leqslant 0. \tag{3.16}$$

From (3.12) and (3.5) we have

$$v - f(u) = (v - au) - (f(u) - au) = (f(\tilde{u}) - a\tilde{u}) - (f(u) - au).$$

Therefore, (3.16) is equivalent to

$$[f(\tilde{u}) - f(u) - a(\tilde{u} - u)](\tilde{u} - u) \leqslant 0.$$

This is satisfied provided

$$\frac{f(\tilde{u}) - f(u)}{\tilde{u} - u} \leqslant a.$$

which is clearly true under the subcharacteristic condition (1.7). $\square$

4. Conclusions

An alternate relaxation system to approximate scalar conservation law in one dimension is proposed. The alteration is proposed so as to have a consistent integral invariant compared to the existing relaxation systems proposed in the literature. L^∞ estimates are established and an appropriate convex entropy function is constructed. From this it can be deduced (on the lines of Natalini) that the entropy solution to the original conservation law can be recovered by letting ε to go to zero.

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